

# Chain Homotopies for Object Topological Representations

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## Abstract

This paper presents a set of tools to compute topological information of simplicial complexes, tools that are applicable to extract topological information from digital pictures. A simplicial complex is encoded in a (non-unique) algebraic-topological format called AM-model. An AM-model for a given object  $K$  is determined by a concrete chain homotopy and it provides, in particular, integer (co)homology generators of  $K$  and representative (co)cycles of these generators. An algorithm for computing an AM-model and the cohomological invariant  $HB1$  (derived from the rank of the cohomology ring) with integer coefficients for a finite simplicial complex in any dimension is designed here, extending the work done in [9] in which the ground ring was a field. A concept of generators which are “nicely” representative is also presented. Moreover, we extend the definition of AM-models to 3D binary digital images and we design algorithms to update the AM-model information after voxel set operations (union, intersection, difference and inverse).

*Keywords:* Simplicial complexes, chain homotopy, cohomology ring, digital topology.

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## 1. Introduction

The problem of adapting topology based methods to discrete data is an active field of research. The set of algorithmic tools for computing topological properties in the setting of digital imagery is relatively small. Mainly, Betti numbers, Euler characteristic, skeletonization and local topological characterization. Recently, homology groups have also been computed in the digital imagery [19].

In order to enlarge this set, an algebraic-topological representation for a given geometric object is developed in [9] in terms of a chain homotopy equivalence (working with coefficients in a field). These models allows to compute the (co)homology groups, representative (co)cycles of the generators, the cup product on cohomology as well as the cohomological number  $HB1$ . Our peculiar approach comes from the following main sources: (a) Eilenberg-Mac Lane work on homology of simplicial sets [5]; (b) Effective Homology Theory [23].

In this paper, we extend the results given in [9] to the integer domain. Algebraic-topological representations of geometric objects are expressed here only in terms of particular chain homotopies (called AM-models). Compared to previous works, these models also encode torsion groups. Moreover, all the algorithms for computing integer homology based on the matrix reduction method to Smith normal form (for example [18, 1, 4, 19]) can be translated into our setting without extra computational cost. Finally, we successfully apply our computational algebraic-topological approach to 3D binary digital images. Moreover, AM-models for 3D binary digital images can be reused after voxel-set operations (union, intersection, difference and inverse).

## 2. Algebraic-topological Models for Integer Homology Computation

In this section, we deal with the concept of AM-model (first given in [8] and [10]) for the computation of all the integer homological information of a simplicial complex. In fact, we redefine this concept in simpler terms, what allows to store the same information in less space.

We first introduce some basic algebraic-topological notions needed throughout the paper, which are extracted, mainly, from [18]. Let  $\mathbf{Z}$  be the ground ring. Considering an ordering on a vertex set  $V$ , a simplex with  $q+1$  affinely independent vertices  $v_0 < \dots < v_q$  of  $V$  is the convex hull of these points,

denoted by  $\sigma^q = \langle v_0, \dots, v_q \rangle$ . The *dimension* of  $\sigma^q$  is  $q$ . If  $i < q$ , an  $i$ -*face* of  $\sigma^q$  is an  $i$ -simplex whose vertices belong to the set  $\{v_0, \dots, v_q\}$ . A *simplicial complex*  $K$  is a collection of simplices such that every face of a simplex of  $K$  is also a simplex of  $K$  and the intersection of any two simplices of  $K$  is either a face of both or empty. The *dimension of*  $K$  is the one of the highest dimensional simplex in  $K$ .  $K^{(q)}$  denotes the set of all the  $q$ -simplices of  $K$ .

A *chain complex*  $\mathcal{C}$  is a sequence

$$\dots \xrightarrow{d_{q+1}} \mathcal{C}_q \xrightarrow{d_q} \mathcal{C}_{q-1} \xrightarrow{d_{q-1}} \dots \xrightarrow{d_1} \mathcal{C}_0 \xrightarrow{d_0} 0.$$

of abelian groups  $\mathcal{C}_q$  (called  $q$ -*chain groups*) and homomorphisms  $d_q : \mathcal{C}_q \rightarrow \mathcal{C}_{q-1}$  (called the *differential* of  $\mathcal{C}$  in dimension  $q$ ), indexed with the integers, such that  $d_{q-1}d_q = 0$  for all  $q$ . In this paper, each group  $\mathcal{C}_q$  is free of finite rank. A chain complex  $\mathcal{C}$  can be encoded as a couple  $(C, d)$ , where: (1)  $C = \{\mathcal{C}_q\}$  and, for each  $q$ ,  $\mathcal{C}_q$  is a base of  $\mathcal{C}_q$ ; (2)  $d = \{d_q\}$  and, for each  $q$ ,  $d_q$  is the differential of  $\mathcal{C}$  in dimension  $q$  with respect to the bases  $\mathcal{C}_q$  and  $\mathcal{C}_{q-1}$ .

Such an algebraic structure can be associated to a given simplicial complex  $K$ , in the following way: a  $q$ -*chain*  $a^q$  is a finite sum  $\sum \lambda_i \sigma_i^q$  where  $\lambda_i \in \mathbf{Z}$  and  $\sigma_i^q \in K^{(q)}$ . The  $q$ -chains form the  $q$ th *chain group* of  $K$ , denoted by  $\mathcal{C}_q(K)$  (with  $q \geq 0$ ). The *boundary* of a  $q$ -simplex  $\sigma^q = \langle v_0, \dots, v_q \rangle$  is the  $(q-1)$ -chain  $\partial_q(\sigma^q) = \sum_{i=0}^q (-1)^i \langle v_0, \dots, \hat{v}_i, \dots, v_q \rangle$ , where  $\hat{v}_i$  means that  $v_i$  is omitted. By linearity,  $\partial_q$  can be extended to  $q$ -chains. Then, the chain complex  $\mathcal{C}(K)$  is the collection of chain groups  $\mathcal{C}_q(K)$  connected by the boundary operators  $\partial_q$ .

Given a chain complex  $\mathcal{C} = (C, d)$ , a  $q$ -chain  $a^q \in \mathcal{C}_q$  is called a  $q$ -*cycle* if  $d_q(a^q) = 0$ . If  $a^q = d_{q+1}(b^{q+1})$  for some  $b^{q+1} \in \mathcal{C}_{q+1}$  then  $a^q$  is called a  $q$ -*boundary*. Denote the groups of  $q$ -cycles and  $q$ -boundaries by  $Z_q$  and  $B_q$  respectively. Define the  $q$ th *homology group* to be the quotient group  $Z_q/B_q$ , denoted by  $\mathcal{H}_q(\mathcal{C})$ . We say that two  $q$ -cycles  $a^q$  and  $b^q$  are *homologous* if there exists a  $(q+1)$ -chain  $c^{q+1}$  such that  $a^q = b^q + d_{q+1}(c^{q+1})$ . For each  $q$ , the integer  $q$ th homology group  $\mathcal{H}_q(\mathcal{C})$  is a finitely generated abelian group isomorphic to  $F_q \oplus T_q$ , where  $F_q = \mathbf{Z} \oplus \dots \oplus \mathbf{Z}$  and  $T_q = \mathbf{Z}/t_1^q \oplus \dots \oplus \mathbf{Z}/t_{m_q}^q$  are the *free subgroup* and the *torsion subgroup* of  $\mathcal{H}_q(\mathcal{C})$ , respectively. The numbers  $t_1^q, \dots, t_{m_q}^q$  which satisfy that  $t_1^q \geq 2$  and  $t_1^q | t_2^q | \dots | t_{m_q}^q$  are called the *torsion coefficients* of  $\mathcal{H}_q(\mathcal{C})$ . The rank of  $F_q$ , denoted by  $\beta_q$ , is called the  $q$ th *Betti number* of  $\mathcal{C}$ . Intuitively,  $\beta_0$  is the number of components of connected pieces,  $\beta_1$  is the number of independent “holes” and  $\beta_2$  is the number of “cavities”. For all  $q$ , there exists a finite number of elements of  $\mathcal{H}_q(\mathcal{C})$  from

which we can deduce all  $\mathcal{H}_q(\mathcal{C})$  elements. These elements are called *homology generators* of dimension  $q$ . We say that  $a^q$  is a *representative  $q$ -cycle* of a homology generator  $\alpha^q$  of dimension  $q$  if  $\alpha^q = a^q + B_q$ . We denote  $\alpha^q = [a^q]$ . Finally, the homology of a simplicial complex  $K$  is defined as the homology of  $\mathcal{C}(K)$ .

As far as the homology computation of a chain complex is concerned, the classical algorithm for computing homology with coefficients in  $\mathbf{Z}$  is the integer reduction algorithm [18]. Given a chain complex  $\mathcal{C} = (C, d)$ , this algorithm consists in reducing the matrix of the boundary operator in each dimension  $q$ , to its Smith Normal Form (SNF), relative to some bases  $\{a_1^q, \dots, a_{m_q}^q\}$  of  $\mathcal{C}_q$  and  $\{e_1^{q-1}, \dots, e_{m_{q-1}}^{q-1}\}$  of  $\mathcal{C}_{q-1}$  such that for some  $t_q, \ell_q, s_{q-1}$  where  $1 \leq s_{q-1} \leq m_{q-1}$  and  $1 \leq t_q \leq \ell_q \leq \min(m_q, s_{q-1})$ ,

- $\partial_{q-1}(e_i^{q-1}) = 0$  for  $1 \leq i \leq s_{q-1}$  and  $\partial_{q-1}(e_i^{q-1}) \neq 0$  for  $s_{q-1} < i \leq m_{q-1}$ .
- $\partial_q(a_i^q) = e_i^{q-1}$ , for  $1 \leq i \leq t_q$ ;
- $\partial_q(a_i^q) = \lambda_i^q e_i^{q-1}$ , where  $\lambda_i^q \in \mathbf{Z}$  and  $\lambda_i^q \geq 2$  for  $t_q < i \leq \ell_q$ ;
- and  $\partial_q(a_i^q) = 0$  for  $\ell_q < i \leq m_q$ .

In this case,

$$F_{q-1} = \mathbf{Z} \oplus \overset{s_{q-1}-\ell_q}{\dots} \oplus \mathbf{Z} \quad \text{and} \quad T_{q-1} = \mathbf{Z}/\lambda_{t_q+1}^q \oplus \dots \oplus \mathbf{Z}/\lambda_{\ell_q}^q.$$

Moreover,  $\{e_{\ell_q+1}^{q-1}, \dots, e_{s_{q-1}}^{q-1}\}$  and  $\{e_{t_q+1}^{q-1}, \dots, e_{\ell_q}^{q-1}\}$  are sets of representative cycles of the generators of  $F_{q-1}$  and  $T_{q-1}$ , respectively.

In [19], an algorithm improving the efficiency of this classical integer reduction algorithm is described. Their technique is mainly based on the results of [4], in which a matrix reduction to integer SNF is determined in an efficient way. We can take advantage of these improvements in the algorithms described here without additional computational cost.

A *chain contraction* [17]  $(f, g, \phi)$  of a chain complex  $\mathcal{C} = (C, d)$  to a chain complex  $\mathcal{C}' = (C', d')$  is a set of three homomorphisms  $f = \{f_q : \mathcal{C}_q \rightarrow \mathcal{C}'_q\}$ ,  $g = \{g_q : \mathcal{C}'_q \rightarrow \mathcal{C}_q\}$  and  $\phi = \{\phi_q : \mathcal{C}_q \rightarrow \mathcal{C}_{q+1}\}$  such that for each  $q$ :

- $f_{q-1}d_q = d'_q f_q$  and  $d_q g_q = g_{q-1} d'_q$ .
- $f_q g_q$  is the identity map of  $\mathcal{C}'_q$ ;

- $\phi_{q-1}d_q + d_{q+1}\phi_q = id_q - g_qf_q$ , that is,  $\phi$  is a chain homotopy of the identity map  $id = \{id_q\}$  of  $\mathcal{C}$  to  $gf$ .

In this case,  $\mathcal{C}'$  has fewer or the same number of generators than  $\mathcal{C}$  while  $\mathcal{C}$  and  $\mathcal{C}'$  have isomorphic homology groups [18, p. 73].

A translation of the integer reduction algorithm in terms of chain contractions has been made in [8, 10]. In those papers an AM-model for a given simplicial complex  $K$  was defined as a chain contraction of  $\mathcal{C}(K)$  to a chain complex  $\mathcal{M}$  such that the SNF,  $A_q$ , of the matrix of the differential of  $\mathcal{M}$  in each dimension  $q$  satisfies that any non-null entry of  $A_q$  is greater than 1. Here we go further and define an AM-model for a given simplicial complex only in terms of the chain homotopy  $\phi$ . We prove that it is possible to recover the other homomorphisms  $f$  and  $g$  and the chain complex  $\mathcal{M}$ . Moreover, our strategy outperforms the previous algorithms for computing integer homology in several points: (1) cohomological features can be computed; (2) we can control the topological changes after addition or deletion of simplices.

**Definition 2.1.** *An AM-model for a simplicial complex  $K$  is a couple  $(C, \phi)$  such that:*

- $C = \{C_q\}$  and, for each  $q$ ,  $C_q$  is a base of  $\mathcal{C}_q(K)$ ;
- $\phi = \{\phi_q\}$  and, for each  $q$ ,  $\phi_q : \mathcal{C}_q(K) \rightarrow \mathcal{C}_{q+1}(K)$  is a homomorphism satisfying that  $\phi_{q+1}\phi_q = 0$  and  $\phi_q\partial_{q+1}\phi_q = \phi_q$ ;
- the chain complex  $\mathcal{M} = (M, d)$  (such that  $M_q = \text{Im } \pi_q$ ,  $\pi_q = id_q - \phi_{q-1}\partial_q - \partial_{q+1}\phi_q$  and  $d_q = \partial_q|_{\mathcal{M}_q}$ ), satisfies, for each  $q$ , that any non-null entry of the SNF of  $d_q$  is greater than 1.

**Theorem 2.1.** *Given an AM-model  $(C, \phi)$  for a simplicial complex  $K$ , we can define a chain contraction  $(f, g, \phi)$  of  $\mathcal{C}(K)$  to a chain complex  $\mathcal{M}$  such that any non-null entry of the SNF of the matrix of the differential of  $\mathcal{M}$ , for each  $q$ , is greater than 1. In particular, if the homology of  $K$  is free then  $\mathcal{M}$  is isomorphic to  $\mathcal{H}(K)$ .*

**Proof.** We only have to define  $f$  as  $\{\pi_q\}$  where, for each  $q$ ,  $\pi_q = id_q - \phi_{q-1}\partial_q - \partial_{q+1}\phi_q$ , and  $g$  as the inclusion. Let us see that  $(f, g, \phi)$  is a chain contraction. It is clear that, for each  $q$ ,  $g_qf_q = id_q - \phi_{q-1}\partial_q - \partial_{q+1}\phi_q$ . On the other hand, let  $a^q \in \mathcal{M}_q$ . There exists  $b^q \in \mathcal{C}_q(K)$  such that  $a^q = \pi_q(b^q) = b^q - \phi_{q-1}\partial_q(b^q) - \partial_{q+1}\phi_q(b^q)$ . Then,  $f_qg_q(a^q) = \pi_q(a^q) = b^q - \phi_{q-1}\partial_q(b^q) -$

$\partial_{q+1}\phi_q(b^q) - \phi_{q-1}\partial_q(b^q) + \phi_{q-1}\partial_q\phi_{q-1}\partial_q(b^q) + \phi_{q-1}\partial_q\partial_{q+1}\phi_q(b^q) - \partial_{q+1}\phi_q(b^q) + \partial_{q+1}\phi_q\phi_{q-1}\partial_q(b^q) + \partial_{q+1}\phi_q\partial_{q+1}\phi_q(b^q) = a^q$ . Second,  $f_{q-1}\partial_q = \partial_q - \partial_q\phi_{q-1}\partial_q = \partial_q f_q$ . Finally, if  $a^q \in \mathcal{M}_q$  then  $\partial_q(a^q) \in \mathcal{M}_q$ . Therefore  $g_{q-1}\partial_q = \partial_q g_q$ .  $\square$

Given a simplicial complex  $K$ , it is possible to define different AM-models for  $K$  since the chain homotopy  $\phi$  and the chain complex  $\mathcal{M}$  are not unique. In the following lemma, we show how to compute a new AM-model from a previous one.

**Lemma 2.1.** *Let  $(C, \phi)$  be an AM-model for a simplicial complex  $K$  and let  $h^q \in \mathcal{C}_q(K)$ .*

- *If  $h^q \in \mathcal{M}_q$  then  $\phi_q(h^q) = 0$ ; and if  $\partial_q(h^q) = 0$  then  $\pi_q(h^q) = h^q$ .*
- *If  $x^q \in \mathcal{C}_q(K)$  and  $h^q \in \mathcal{M}_q$  such that  $\partial_q(x^q) = 0$  and  $\pi_q(x^q) = h^q$  then  $[x^q] = [h^q]$  and we can define a new AM-model  $(C, \phi')$  as  $\phi'_q(x^q) := \phi_q(h^q)$ ,  $\phi'_q(h^q) := \phi_q(x^q)$  and  $\phi' := \phi$  for the rest. In this case,  $\pi'_q(x^q) = (id_q - \phi'_{q-1}\partial_q - \partial_{q+1}\phi'_q)(x^q) = x^q$  and  $\pi'_q(h^q) = x^q$ .*

**Algorithm 2.2.** *Computing an AM-model for a Finite Simplicial Complex.*

INPUT: A simplicial complex  $K$  of dimension  $n$ .

For  $q = 1$  to  $q = n$  do

reduce the matrix of  $\partial_q$  to its SNF relative to some bases  $\{a_1^q, \dots, a_{m_q}^q\}$  of  $\mathcal{C}_q(K)$  and  $\{e_1^{q-1}, \dots, e_{m_{q-1}}^{q-1}\}$  of  $\mathcal{C}_{q-1}(K)$  such that for some  $s_{q-1}$  where  $1 \leq s_{q-1} \leq m_{q-1}$ :

$\partial_{q-1}(e_i^{q-1}) = 0$ , for  $1 \leq i \leq s_{q-1}$   
and  $\partial_{q-1}(e_i^{q-1}) \neq 0$ , for  $s_{q-1} < i \leq m_{q-1}$ ;

for some  $t_q, \ell_q$  where  $1 \leq t_q \leq \ell_q \leq \min(m_q, s_{q-1})$ :

$\partial_q(a_i^q) = e_i^{q-1}$ , for  $1 \leq i \leq t_q$ ;  
 $\partial_q(a_i^q) = \lambda_i^q e_i^{q-1}$ ,  $\lambda_i^q \in \mathbf{Z}$ ,  $\lambda_i^q \geq 2$  for  $t_q < i \leq \ell_q$ ;  
and  $\partial_q(a_i^q) = 0$  for  $\ell_q < i \leq m_q$ .

Define  $C_{q-1} := \{e_1^{q-1}, \dots, e_{m_{q-1}}^{q-1}\}$ ,  $C_q := \{a_1^q, \dots, a_{m_q}^q\}$ ,

$\phi_{q-1}(e_i^{q-1}) := a_i^q$  for  $1 \leq i \leq t_q$ ,  
 $\phi_{q-1}(e_i^{q-1}) := 0$  for  $t_q < i \leq s_{q-1}$   
and  $\phi_q(a_i^q) := 0$  for  $1 \leq i \leq t_q$ .

OUTPUT: The couple  $(C, \phi)$ .

Notice that, for  $q = k$ ,  $\phi_k(a_i^k)$  is defined for  $1 \leq i \leq t_k$ . After,  $\phi_k$  is again defined when  $q = k + 1$ , but not for the previous elements  $a_i^k$ , since  $a_i^k \notin \text{Im } \partial_{k+1}$ , for  $1 \leq i \leq t_k$ .

**Theorem 2.2.** *Let  $K$  be a finite simplicial complex of dimension  $n$ . The output of Algorithm 2.2,  $(C, \phi)$ , defines an AM-model for  $K$ . Moreover, integer homology generators and representative cycles of these generators can be directly obtained from  $\mathcal{M}$ .*

**Proof.** First of all, let us see that for each  $q$ , if  $x^q \in C_q$ , then  $\pi_q(x^q) = 0$  or  $\pi_q(x^q) = x^q$ , where  $\pi_q = id_q - \phi_{q-1}\partial_q - \partial_{q+1}\phi_q$ . Since the matrix of the differential in dimension  $q$  with respect to the base  $C_q$  coincides with its SNF, then for each  $x^q \in C_q$ :

- If  $\partial_q(x^q) = y^{q-1}$  for some  $y^{q-1} \in C_{q-1}$ , then  $\phi_{q-1}(y^{q-1}) = x^q$  and  $\phi_q(x^q) = 0$ . Therefore  $\pi_q(x^q) = 0$ .
- If  $\partial_q(x^q) = \lambda y^{q-1}$ , for some  $y^{q-1} \in C_{q-1}$  and  $\lambda \in \mathbf{Z}$ ,  $\lambda \geq 2$ , then  $\phi_{q-1}(y^{q-1}) = 0$  and  $\phi_q(x^q) = 0$ . Therefore,  $\pi_q(x^q) = x^q$ .
- If  $\partial_q(x^q) = 0$  and there exist  $z^{q+1} \in C_{q+1}$  such that  $\partial_{q+1}(z^{q+1}) = x^q$  then  $\phi_q(x^q) = z^{q+1}$  and  $\pi_q(x^q) = 0$ .
- If  $\partial_q(x^q) = 0$  and there exist  $z^{q+1} \in C_{q+1}$  and  $\lambda \in \mathbf{Z}$ ,  $\lambda \geq 2$ , such that  $\partial_{q+1}(z^{q+1}) = \lambda x^q$ , then  $\phi_q(x^q) = 0$  and  $\pi_q(x^q) = x^q$ .
- If  $\partial_q(x^q) = 0$  and there is neither  $\lambda \in \mathbf{Z}$ ,  $\lambda \neq 0$ , nor  $z^{q+1} \in C_{q+1}$ , such that  $\partial_{q+1}(z^{q+1}) = \lambda x^q$ , then  $\phi_q(x^q) = 0$  and  $\pi_q(x^q) = x^q$ .

Therefore, a base of  $\mathcal{M}$  in each dimension  $q$  is the set  $M_q = \{x^q : x^q \in C_q \text{ and } \pi_q(x^q) = x^q\}$ . Now, for each  $q$ , let  $\{x_1^q, \dots, x_{m_q}^q\}$  be the elements of  $M_q$  and  $\{y_1^{q-1}, \dots, y_{m_{q-1}}^{q-1}\}$  the elements of  $M_{q-1}$ . For some  $s_{q-1}$ ,  $1 \leq s_{q-1} \leq m_{q-1}$ ,  $\partial_{q-1}(y_i^{q-1}) = 0$  for  $1 \leq i \leq s_{q-1}$  and  $\partial_{q-1}(y_i^{q-1}) \neq 0$  for  $s_{q-1} < i \leq m_{q-1}$ . For some  $\ell_q$  where  $1 \leq \ell_q \leq \min(m_q, s_{q-1})$ ,

- $\partial_q(x_i^q) = \lambda_i^q y_i^{q-1}$ , where  $\lambda_i^q \in \mathbf{Z}$  and  $\lambda_i^q \geq 2$  for  $1 \leq i \leq \ell_q$ ;
- $\partial_q(x_i^q) = 0$  for  $\ell_q < i \leq m_q$ .

In this case,

$$F_{q-1} = \mathbf{Z} \oplus \overset{s_{q-1}-\ell_q}{\dots} \oplus \mathbf{Z} \quad \text{and} \quad T_{q-1} = \mathbf{Z}/\lambda_1^q \oplus \dots \oplus \mathbf{Z}/\lambda_{\ell_q}^q.$$

Moreover,  $\{y_{\ell_q+1}^{q-1}, \dots, y_{s_{q-1}}^{q-1}\}$  and  $\{y_1^{q-1}, \dots, y_{\ell_q}^{q-1}\}$  are sets of representative cycles of the generators of  $F_{q-1}$  and  $T_{q-1}$ , respectively.  $\square$

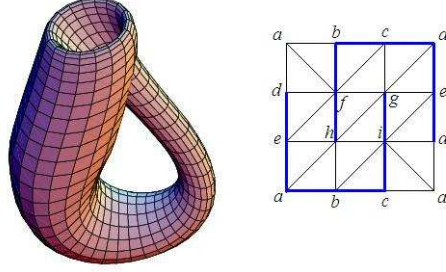


Figure 1: The Klein bottle and a triangulation of it.

**Example 2.3.** Consider the simplicial complex  $K$  in Figure 1 whose underlying space is the Klein bottle [18, p. 283]. Applying Algorithm 2.2, we obtain that the vertex  $\langle a \rangle$  is an element of  $C_0$ . The rest of the elements of  $C_0$  are the boundaries of the 1-simplices marked in blue in Figure 1. These 1-simplices are also elements of  $C_1$ . Denote by  $x$  one of these 1-simplices. The cycles  $\alpha_1 := \langle a, d \rangle + \langle d, e \rangle - \langle a, e \rangle$  and  $\alpha_2 := \langle a, b \rangle + \langle b, c \rangle - \langle a, c \rangle$  are also elements of  $C_1$ . The rest of the elements of  $C_1$  are the boundaries of all the 2-simplices except for  $\langle f, g, h \rangle$ . These 2-simplices belong to  $C_2$ . Denote by  $y$  one of these 2-simplices. The 2-chain consisting in the sum of all the triangles in  $K$ ,  $\gamma := -\langle a, b, f \rangle - \langle b, c, f \rangle + \langle a, c, g \rangle - \langle a, e, g \rangle + \langle e, g, i \rangle - \langle e, d, i \rangle + \langle c, d, i \rangle - \langle a, c, d \rangle + \langle b, c, i \rangle + \langle a, b, h \rangle - \langle a, e, h \rangle - \langle e, d, f \rangle + \langle a, d, f \rangle + \langle c, f, g \rangle - \langle f, g, h \rangle - \langle h, g, i \rangle - \langle b, h, i \rangle$ , is also an element of  $C_2$ . The image of  $\partial$ ,  $\phi$  and  $\pi$  on  $C$  and the SNF of the matrix of the differential in dimension 1 and 2 are given below:

$C$	$\langle a \rangle$	$\partial_1(x)$	$x$	$\alpha_1$	$\alpha_2$	$\partial_2(y)$	$y$	$\gamma$
$\partial$	0	0	$\partial_1(x)$	0	0	0	$\partial_2(y)$	$2\alpha_1$
$\phi$	0	$x$	0	0	0	$y$	0	0
$\pi$	$\langle a \rangle$	0	0	$\alpha_1$	$\alpha_2$	0	0	$\gamma$

$\partial_1$	$x$	$\partial_2(y)$	$\alpha_1$	$\alpha_2$
$\partial_1(x)$	1	0	0	0
$\langle a \rangle$	0	0	0	0

$\partial_2$	$y$	$\gamma$
$\partial_2(y)$	1	0
$\alpha_1$	0	2
$\alpha_2$	0	0

Therefore,  $M_0 = \{\langle a \rangle\}$ ,  $M_1 = \{\alpha_1, \alpha_2\}$  and  $M_2 = \{\gamma\}$ ,  $\partial|_{\mathcal{M}_0}(\langle a \rangle) = 0$ ,  $\partial|_{\mathcal{M}_1}(\alpha_1) = 0$ ,  $\partial|_{\mathcal{M}_1}(\alpha_2) = 0$  and  $\partial|_{\mathcal{M}_2}(\gamma) = 2\alpha_1$ . We obtain that  $\mathcal{H}_0(K) \simeq \mathbf{Z}$ ,  $\mathcal{H}_1(K) \simeq \mathbf{Z} \oplus \mathbf{Z}/2$  and representative cycles of the homology generators are  $\langle a \rangle$  for  $\mathcal{H}_0(K)$ ,  $\alpha_2$  for the free part of  $\mathcal{H}_1(K)$  and  $\alpha_1$  for the torsion part.



### 3. Extracting Integer Cohomology Information from AM-models

In this section, we extend the work done in [9] (with coefficients in a field) for computing cohomology (the dual notion of homology) over the integer domain. Working with coefficients in a field, homology groups are free and isomorphic to cohomology groups. Nevertheless, integer homology and cohomology can have a torsion part and they are not isomorphic, in general. The cohomology groups have an additional multiplicative structure, the cup product, from which we can derive finer invariants than homology.

Let  $\mathcal{C} = (C, d)$  be a chain complex. The *cochain complex*  $\mathcal{C}^* = (C^*, \delta)$  in each dimension  $q$  is the group of  $q$ -cochains with coefficients in  $\mathbf{Z}$ ,  $\mathcal{C}^q = \{c : C_q \rightarrow \mathbf{Z} \text{ such that } c \text{ is a homomorphism}\}$ . If  $C_q = \{a_1, \dots, a_{m_q}\}$  is a base of  $C_q$  then a base of  $\mathcal{C}^q$  is  $\mathcal{C}^q = \{a_1^*, \dots, a_{m_q}^*\}$ , where  $a_i^* : C_q \rightarrow \mathbf{Z}$  is given by  $a_i^*(a_i) = 1$  and  $a_i^*(a_j) = 0$  for  $1 \leq i, j \leq m_q$  and  $j \neq i$ . For all  $q$ , the differential  $d_{q+1}$  on  $\mathcal{C}_{q+1}$  induces the *codifferential*  $\delta^q : \mathcal{C}^q \rightarrow \mathcal{C}^{q+1}$  via  $\delta^q(c) = cd_{q+1}$ , so that  $\delta^q$  increases the dimension by one. Define  $Z^q$  to be the kernel of  $\delta^q$  and  $B^{q+1}$  to be its image. These groups are called the group of  $q$ -cocycles and  $q+1$ -coboundaries, respectively. Define the  $q$ th cohomology group,  $\mathcal{H}^q(\mathcal{C}) = Z^q/B^q$  for  $q \geq 0$ . For each  $q$ , the integer  $q$ th cohomology group  $\mathcal{H}^q(\mathcal{C})$  is a finitely generated abelian group isomorphic to  $F^q \oplus T^q$ , where  $F^q$  and  $T^q$  are the *free subgroup* and the *torsion subgroup* of  $\mathcal{H}^q(\mathcal{C})$ , respectively. The rank of  $F^q$  coincides with the rank of  $F_q$  for each  $q$ .

**Theorem 3.1.** *Let  $(C, \phi)$  be the output of Algorithm 2.2 for a given finite simplicial complex  $K$ . The integer cohomology of  $K$  and representative cocycles of integer cohomology generators can be directly obtained from  $\mathcal{M} = \text{im } \pi$ , where  $\pi = \text{id} - \phi\partial - \partial\phi$ .*

**Proof.** A base of  $\mathcal{M}$  is the set  $M = \{x : x \in C \text{ and } \pi(x) = x\}$ . Now, for each  $q$ , let  $\{x_1, \dots, x_{m_q}\}$  be the elements of  $M_q$  and  $\{y_1, \dots, y_{m_{q-1}}\}$  the elements of  $M_{q-1}$ . For some  $s_{q-1}$ ,  $1 \leq s_{q-1} \leq m_{q-1}$ ,  $\partial_{q-1}(y_i) = 0$  for  $1 \leq i \leq s_{q-1}$  and  $\partial_{q-1}(y_i) \neq 0$  for  $s_{q-1} < i \leq m_{q-1}$ . For some  $\ell_q$  where  $1 \leq \ell_q \leq \min(m_q, s_{q-1})$ ,  $\partial_q(x_i) = \lambda_i y_i$ , where  $\lambda_i \in \mathbf{Z}$  and  $\lambda_i \geq 2$  for  $1 \leq i \leq \ell_q$ ; and  $\partial_q(x_i) = 0$  for  $\ell_q < i \leq m_q$ . In this case,

$$F^{q-1} = \mathbf{Z} \oplus \overset{s_{q-1}-\ell_q}{\dots} \oplus \mathbf{Z} \quad \text{and} \quad T^q = \mathbf{Z}/\lambda_1 \oplus \dots \oplus \mathbf{Z}/\lambda_{\ell_q}.$$

Moreover,  $\{y_{\ell_q+1}^*, \dots, y_{s_{q-1}}^*\}$  and  $\{x_1^*, \dots, x_{\ell_q}^*\}$  are sets of representative cocycles of the generators of  $F^{q-1}$  and  $T^q$ , respectively.  $\square$

**Example 3.1.** Consider the AM-model  $(C, \phi)$  obtained in Example 2.3 for the simplicial complex  $K$  whose underlying space is the Klein bottle. Starting from the chain complex  $\mathcal{M}$  whose base is  $\{\langle a \rangle, \alpha_1, \alpha_2, \gamma\}$  and differential  $\partial|_{\mathcal{M}}$ , we construct in a straightforward way the cochain complex  $\mathcal{M}^*$  whose base is  $\{\langle a \rangle^*, \alpha_1^*, \alpha_2^*, \gamma^*\}$  and codifferential  $\delta$  given by:  $\delta_0(\langle a \rangle^*) = \langle a \rangle^* \partial|_{\mathcal{M}_1} = 0$ ,  $\delta_1(\alpha_1^*) = \alpha_1^* \partial|_{\mathcal{M}_2} = 0$ ,  $\delta_1(\alpha_2^*) = \alpha_2^* \partial|_{\mathcal{M}_2} = 2\gamma^*$ ,  $\delta_2(\gamma^*) = \gamma^* \partial|_{\mathcal{M}_3} = 0$ .

Therefore we obtain that  $\mathcal{H}^0(K) \simeq \mathbf{Z}$ ,  $\mathcal{H}^1(K) \simeq \mathbf{Z}$  and  $\mathcal{H}^2(K) \simeq \mathbf{Z}/2$ ; and the representative cocycles are:  $\langle a \rangle^*$  in dimension 0,  $\alpha_1^*$  in dimension 1 and  $\gamma^*$  in dimension 2.

The cochain complex  $\mathcal{C}^*(K)$  is a ring with the cup product  $\smile: \mathcal{C}^p(K) \times \mathcal{C}^q(K) \rightarrow \mathcal{C}^{p+q}(K)$  given by:

$$(c \smile c')(\langle v_0, \dots, v_{p+q} \rangle) = c(\langle v_0, \dots, v_p \rangle) \cdot c'(\langle v_{p+1}, \dots, v_{p+q} \rangle).$$

This product induces an operation  $\smile: \mathcal{H}^p(K) \times \mathcal{H}^q(K) \rightarrow \mathcal{H}^{p+q}(K)$ , via  $[c] \smile [c'] = [c \smile c']$ , that is bilinear, associative, commutative up to a sign, independent of the ordering of the vertices of  $K$  and homotopy-type invariant [18, p. 289].

Working with coefficients in  $\mathbf{Z}/2$ , a new cohomology invariant called *HB1* is obtained in [9]. The idea is to put into a matrix form the multiplication table of the cup product of cohomology generators of dimension 1. The following algorithm compute *HB1* working with integer coefficients. Assume that an AM-model for  $K$ ,  $(C, \phi)$ , is computed using Algorithm 2.2. Then, for each  $q$ ,  $M_q = \{x : x \in C_q \text{ and } \pi_q(x) = x\}$  is a base of  $\mathcal{M}_q = \text{im } \pi_q$ , where  $\pi_q = id_q - \phi_q \partial_q - \partial_q \phi_q$ . Let  $M^q = \{x^* : x \in M_q\}$  where  $x^* : \mathcal{M}_q \rightarrow \mathbf{Z}$  is such that  $x^*(x) = 1$  and  $x^*(z) = 0$  for  $z \in M_q$  and  $z \neq x$ . Suppose that there are  $r_i$  elements in  $C_i$  and  $s_i$  cocycles in  $M^i$ ,  $i = 1, 2$ . Then the following algorithm computes the cohomological invariant *HB1*, working with integer coefficients, in  $\mathcal{O}(r_1^2 s_1^2 r_2 s_2)$ .

**Algorithm 3.2.** Algorithm for computing the cohomological invariant *HB1*.

INPUT: An AM-model  $(C, \phi)$  for a simplicial complex  $K$  computed using Algorithm 2.2.

Let  $M^* = \{x^* : x \in C \text{ and } \pi(x) = x\}$ .

Let  $\{\alpha_1^*, \dots, \alpha_p^*\}$  and  $\{\gamma_1^*, \dots, \gamma_m^*\}$  be the sets of 1 and 2-cocycles in  $M^*$ , respectively.

For  $i = 1$  to  $p$  do

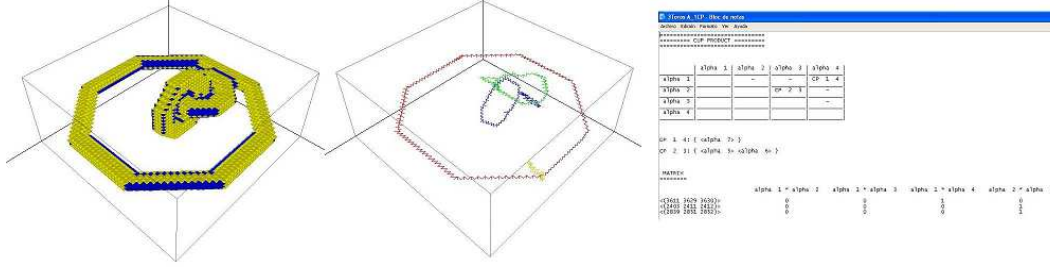


Figure 2: The simplicial complex  $T$ , representative cocycles of the generators of  $\mathcal{H}^1(T)$  and the multiplication table of the cup product.

```

For  $j = i$  to  $p$  do
  For  $k = 1$  to  $m$  do
     $b_{((i,j),k)} := (\alpha_i^* \smile \alpha_j^*)(\gamma_k)$ .
HB1 := the rank of the 2D matrix of integers  $(b_{((i,j),k)})$ .
OUTPUT: The integer HB1.

```

The implementation of the algorithm described above has already been made [11]. We have tested it on several 3D objects. We give here an example of the computation of the cohomology, representative cocycles of cohomology generators and the invariant  $HB1$ .

**Example 3.3.** Consider the simplicial complex  $T$  whose underlying space is showed in Figure 2 (on the left). It consists in 11847 simplices. The running time for computing an AM-model for  $T$  and the homology of  $T$  using a Pentium 4, 3.2 GHz, 1Gb RAM is 2 seconds. We obtain that  $\beta_0 = 1$ ,  $\beta_1 = 4$  and  $\beta_2 = 3$ . The running time for computing the cup product is 1.5 seconds. In Figure 2 (on the center), the 1 and 2-simplices, on which the representative cocycles are non-null, are drawn. The table on the right of Figure 2 shows the results of the cup product of any two cohomology generators of dimension 1. Finally,  $HB1 = 2$ .

#### 4. AM-models for 3D Digital Images

An important issue in Digital Volume Processing is to design efficient algorithms for analysis and processing in grids such as the face-centered cubic (fcc) and the body-centered cubic (bcc) grids [12], since it is very easy to

obtain data structures for them. The bcc and the fcc grid are the generalizations to 3D of the two-dimensional hexagonal grid. In the bcc grid, the voxels consist of truncated octahedra, and in the fcc grid, the voxels consist of rhombic dodecahedra. They are better approximations of Euclidean balls than the cube.

A 3D digital image  $\mathcal{I}$  is encoded as a tuple  $(V, I, b, w)$ , where  $V$  is the set of grid points in a 3D grid,  $I$  is the set of black points and  $b$  (resp.  $w$ ) determines the neighborhood relations between black points (resp. white points) in the grid. A bcc grid is equivalent to a grid  $\mathcal{V}$  in which the grid points are those  $(x_1, x_2, x_3) \in \mathbf{Z}^3$  such that  $x_1 \equiv x_2 \equiv x_3 \pmod{2}$  (see [15]). The only Voronoi adjacency relation on  $\mathcal{V}$  is the 14-adjacency. Using this adjacency, it is straightforward to associate to a digital image  $\mathcal{I} = (\mathcal{V}, I, 14, 14)$  a unique simplicial complex  $K(I)$  (up to isomorphism) with the same topological information as  $I$ . It is called the *simplicial representation of  $I$* . The  $i$ -simplices of  $K(I)$  ( $i \in \{0, 1, 2, 3\}$ ) are constituted by the different sets of  $i$  mutually 14-neighbor points in  $I$ . Since an isomorphism of digital images is equivalent to a simplicial homeomorphism of the corresponding simplicial representations, we define the (co)homology of  $I$  as the (co)homology of  $K(I)$ . Moreover, we define an *AM-model* for  $\mathcal{I}$  as an AM-model for its simplicial representation  $K(I)$ ,  $(C_I, \phi_I)$ , where  $C_I$  is a base for  $\mathcal{C}(K(I))$ . Therefore, the simplicial complexes considered in this section are embedded in  $\mathbf{R}^3$ , then their homology groups vanish for dimensions greater than 3 and they are torsion-free for dimensions 0, 1 and 2 (see [2, ch.10]). Moreover, the value of all the possible non-null entries of the SNF of the matrix of the differential of  $\mathcal{C}(K)$  (where  $K$  is a simplicial complex embedded in  $\mathbf{R}^3$ ) in each dimension must be 1; and an AM-model  $(C, \phi)$  for  $K$  satisfies that the chain complex  $\mathcal{M} = \text{im } \pi = id - \phi\partial - \partial\phi$  is isomorphic to the homology of  $K$ .

All the algorithms explained in this paper have been implemented [11] using as a grid the set of points with integer coordinates in the Euclidean 3-space  $\mathbf{Z}^3$  and the 14-adjacency by which the neighbors of a grid point (black or white) with integer coordinates  $(x_1, x_2, x_3)$  are:  $(x_1 \pm 1, x_2, x_3)$ ,  $(x_1, x_2 \pm 1, x_3)$ ,  $(x_1, x_2, x_3 \pm 1)$ ,  $(x_1 + 1, x_2 - 1, x_3)$ ,  $(x_1 - 1, x_2 + 1, x_3)$ ,  $(x_1 + 1, x_2, x_3 - 1)$ ,  $(x_1 - 1, x_2, x_3 + 1)$ ,  $(x_1, x_2 + 1, x_3 - 1)$ ,  $(x_1, x_2 - 1, x_3 + 1)$ ,  $(x_1 + 1, x_2 + 1, x_3 - 1)$ ,  $(x_1 - 1, x_2 - 1, x_3 + 1)$ . The digital spaces  $(\mathbf{Z}^3, 14, 14)$  and  $(\mathcal{V}, 14, 14)$  are isomorphic: a grid point  $(x_1, x_2, x_3)$  of  $(\mathbf{Z}^3, 14, 14)$  can be associated to the point  $(x_1 + x_2 + 2x_3, -x_1 + x_2, -x_1 - x_2)$  of  $(\mathcal{V}, 14, 14)$ .

In order to compute an AM-model for a digital image  $\mathcal{I} = (\mathbf{Z}^3, I, 14, 14)$ , we take advantage of the particular structure of  $K(I)$  in the way that we

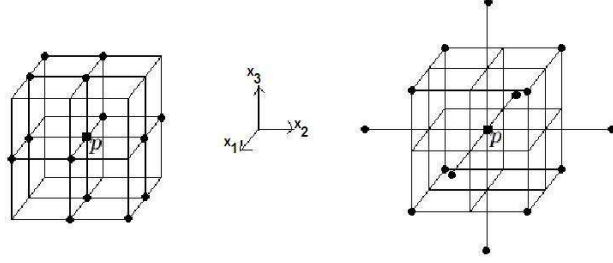


Figure 3: The 14-neighbors of a point  $p \in \mathbb{Z}^3$  (on the left) and  $p \in \mathcal{V}$  (on the right).

consider as the input of Algorithm 2.2 the following special initial base:

- In dimension 0, it consists in the set of the vertices of  $K(I)$ , except for the vertices  $\langle (x_1, x_2, x_3) \rangle$  such that  $x_3$  is odd, which are replaced by the boundary of the 1-chains  $a = \langle (x_1, x_2, x_3 - 1), (x_1, x_2, x_3) \rangle$ , if  $a \in K(I)$ .
- In dimension 1, it consists in the set of all the edges in  $K(I)$ , except for the edges of the form  $\langle (x_1, x_2, x_3), (x_1, x_2 + 1, x_3) \rangle$ , which are replaced by the boundary of the 2-chains  $b = \langle (x_1, x_2, x_3), (x_1, x_2, x_3 + 1), (x_1, x_2 + 1, x_3) \rangle$  if  $b \in K(I)$ .
- In dimension 2, it consists in the set of all the triangles in  $K(I)$ , except for the triangles of the form  $\langle (x_1 - 1, x_2 + 1, x_3), (x_1, x_2, x_3), (x_1, x_2 + 1, x_3) \rangle$ , which are replaced by the boundary of the 3-chains  $c = \langle (x_1 - 1, x_2, x_3 + 1), (x_1 - 1, x_2 + 1, x_3), (x_1, x_2, x_3), (x_1, x_2 + 1, x_3) \rangle$ , if  $c \in K(I)$ .
- In dimension 3, it consists in the set of all the tetrahedra in  $K(I)$ .

We then reduce the matrix of  $\partial_q$  relative to this base to its SNF and it holds that we do not have to modify the rows and columns corresponding to the chains  $a, b, c, \partial(a), \partial(b)$  and  $\partial(c)$ .

In the following table we present the running time for computing AM-models for the 3D digital images showed in Figure 4 using the program developed in [11].

Image $\mathcal{I}$	Number of points of $I$	Time for computing	$\beta_0$	$\beta_1$	$\beta_2$
$I_1$	26308	50 seconds	2	9	3
$I_2$	31012	38 seconds	138	419	13
$I_3$	18842	27 seconds	1	277	5

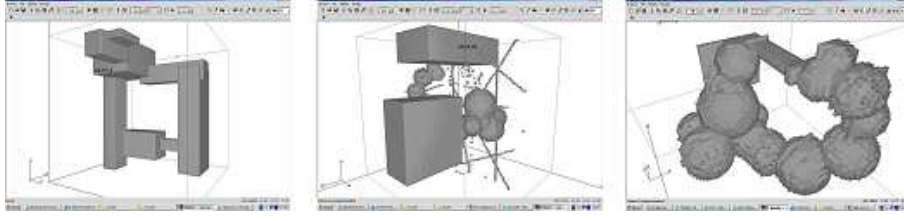


Figure 4: The 3D digital images  $I_1$ ,  $I_2$  and  $I_3$ .

#### 4.1. Computing “Good” Representative Cycles of Homology Generators

In [6], algorithms for obtaining “optimal” generators of the first homology group are developed using Dijkstra’s shortest path algorithm for any oriented 2-manifolds. The algorithms presented in [20] produce “nice” representative cycles of homology generators that always fit on the boundaries of the image.

We sketch here some techniques for drawing “good” representative cycles of homology generators in the context of digital volumes. Let  $(C_I, \phi_I)$  be an AM-model for a 3D digital image  $\mathcal{I} = (\mathcal{V}, I, 14, 14)$ . We denote by  $\partial\mathcal{I} = (\mathcal{V}, \partial I, 14, 14)$  the digital image such that  $\partial I$  is the set of points of  $I$  with at least one 14-neighbor white point of  $\mathcal{I}$ . We say that  $M$  is a set of “good” representative cycles of homology generators of  $I$  if each chain of  $M$  satisfies that:

- it is a cycle;
- it belongs to  $\mathcal{C}(K(\partial I))$ ;
- in dimension 0, it is a vertex;
- in dimension 1, it is an *elementary cycle* (that is, it is connected, each vertex is shared by exactly two edges and two consecutive edges can not belong to the same triangle in  $K(I)$ );
- in dimension 2, it is an *elementary cavity* (that is, it is a connected 2-cycle with exactly one white connected component inside and three triangles can not belong to the same tetrahedra in  $K(I)$ ).

First of all, suppose we have an AM-model for  $\partial\mathcal{I}$ ,  $(C_{\partial I}, \phi_{\partial I})$ , and  $\mathcal{I}$ ,  $(C_I, \phi_I)$ . Let  $\pi_{\partial I} = id - \partial\phi_{\partial I} - \phi_{\partial I}\partial$  and  $\pi_I = id - \partial\phi_I - \phi_I\partial$ . Let us denote by  $\{\alpha_1, \dots, \alpha_n\}$  the elements of  $M_{\partial I}$  which is a base of  $\mathcal{M}_{\partial I} = \text{im } \pi_{\partial I}$ . Suppose

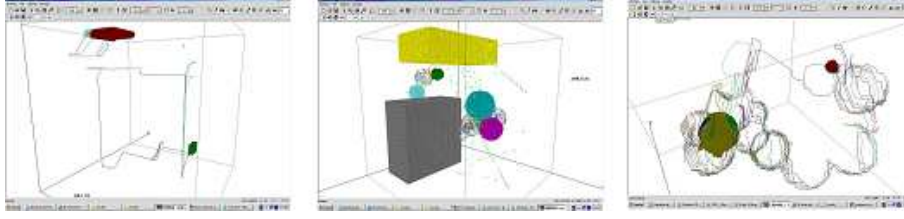


Figure 5: Representative cycles of the homology generators of the images  $I_1$ ,  $I_2$  and  $I_3$ .

that if  $h \in M_{\partial I}$  then  $\partial(h) = 0$  (we can obtain this if we compute the AM-model for  $\partial I$  using Algorithm 2.2). The cycles of  $M_{\partial I}$  are representative cycles of the homology generators of  $\partial I$ . Decompose and replace each 0-cycle in  $M_{\partial I}$  by its constitutive vertices, each 1-cycle in  $M_{\partial I}$  by its elementary cycles and each 2-cycle in  $M_{\partial I}$  by its elementary cavities. Note that all the homology generators of  $I$  are homology generators of  $\partial I$ , therefore a set of representative cycles of the homology generators of  $I$  is a subset of  $M_{\partial I}$ . Let  $M'_I$  be the set  $\{\alpha : \alpha \in M_{\partial I} \text{ and } \pi_I(\alpha) = \alpha\}$ . Obtain the new AM-model  $(C_I, \phi'_I)$  applying Lemma 2.1 to the elements of the set  $M'_I$ . Then,  $(C_I, \phi'_I)$  is a new AM-model for  $\mathcal{I}$ ; and  $M'_I$  is a base of  $\mathcal{M}_I = \text{im } \pi_I$  and a set of good representative cycles of homology generators of  $I$ .

#### 4.2. AM-models after Adding or Deleting a Voxel

Now, we study the problem of topologically controlling a digital image using AM-models when it suffers local changes. More concretely, we show how to compute an AM-model for a digital image when a voxel is added or deleted using the AM-model computed before. Adding or deleting a voxel  $v$  of  $\mathcal{I}$  means to change the color of a point  $v$  in  $\mathcal{I}$  and it consists of adding or deleting a set of simplices having  $v$  as a vertex. Since we work with simplicial complexes representing 3D digital images considering the 14-adjacency, the maximum number of simplices having  $v$  as a vertex is 74. The key idea of both algorithms is that when a  $q$ -simplex  $\sigma$  is added to or deleted from an AM-model for a simplicial complex  $K$ , we only have to put into the SNF the column of the matrix of  $\partial_q$  relative to  $\sigma$  to obtain the new AM-model.

**AM-models after Adding a Voxel.** Given a digital image  $\mathcal{I} = (\mathcal{V}, I, 14, 14)$ , we add to  $I$  a point  $v \notin I$  to obtain a new digital image denoted by  $\mathcal{I}^{\cup v} = (\mathcal{V}, I \cup \{v\}, 14, 14)$ . Therefore, the addition of a point  $v$  to  $I$  consists in the addition to  $K(I)$  of all the simplices of  $K(I \cup \{v\})$  having  $v$  as a vertex. In each step of the process, one simplex is added. Given an AM-



model for a digital image  $\mathcal{I} = (\mathcal{V}, I, 14, 14)$  such that  $I$  has  $m$  points, the following algorithm computes an AM-model for  $\mathcal{I}^{\cup v}$ , with integer coefficients, in  $\mathcal{O}(m^2)$

**Algorithm 4.1.** *An Incremental Algorithm for Computing an AM-model for a 3D Digital Image  $\mathcal{I}$ .*

INPUT: A digital image  $(\mathcal{V}, I, 14, 14)$ , a point  $v \notin I$  and an AM-model  $(C_I, \phi_I)$  for  $\mathcal{I}$  such that, in each dimension  $q$ , the matrix of  $\partial_q$  with respect to  $C_I$  coincides with its SNF.  
 Let  $\{\sigma_1, \dots, \sigma_n\}$  ( $n \leq 74$ ) be the ordered-by-increasing-dimension set of all the simplices of  $K(I \cup \{v\})$  having  $v$  as a vertex.  
 $C^{\cup v} := C_I$  and  $\phi^{\cup v} := \phi_I$ .  
 For  $i = 1$  to  $i = n$  do:  
   Let  $q$  be the dimension of  $\sigma_i$ , let  $C_q^{\cup v} = \{a_1, \dots, a_r\}$  and  $C_{q-1}^{\cup v} = \{e_1, \dots, e_s\}$  such that  $\partial_q(a_j) = e_j$  for  $1 \leq j \leq t$ ,  $\partial_q(a_j) = 0$  for  $t < j \leq r$  and  $\partial_q(\sigma_i) = \sum_{\ell=1}^s \rho_\ell e_\ell$  for some  $\rho_\ell \in \mathbb{Z}$ .  
    $a := \sigma_i - \sum_{\ell=1}^t \rho_\ell a_\ell$  and  $C_q^{\cup v} := \{a_1, \dots, a_r, a\}$ .  
   If  $\rho_\ell = 0$  for  $\ell > t$  then  $\phi^{\cup v}(a) := 0$ .  
   Else obtain the SNF of the matrix of  $\partial_q$  relative to some base  $C_{q-1}^{\cup v} := \{e_1, \dots, e_t, e'_{t+1}, \dots, e'_s\}$  then  $\phi^{\cup v}(a) := 0$ ,  $\phi^{\cup v}(e'_{t+1}) := a$  and  $\phi^{\cup v}(e'_j) := 0$  for the rest.  
 OUTPUT: An AM-model  $(C^{\cup v}, \phi^{\cup v})$  for  $\mathcal{I}^{\cup v}$ .

**AM-models after Deleting a Voxel.** Given a digital image  $\mathcal{I} = (\mathcal{V}, I, 14, 14)$ , we delete from  $I$  a point  $v \in I$  to obtain a new digital image denoted by  $\mathcal{I}^{\setminus v} = (\mathcal{V}, I \setminus \{v\}, 14, 14)$ . Therefore, the deletion of a point  $v$  from  $I$  consists in the deletion from  $K(I)$  of all the simplices of  $K(I)$  having  $v$  as a vertex. In each step of the process, one simplex is deleted. Given an AM-model for a digital image  $\mathcal{I} = (\mathcal{V}, I, 14, 14)$  such that  $I$  has  $m$  points, the following algorithm computes an AM-model for  $\mathcal{I}^{\setminus v}$ , with integer coefficients, in  $\mathcal{O}(m^2)$

**Algorithm 4.2.** *A Decremental Algorithm for Computing an AM-model for a 3D Digital Image  $\mathcal{I}$ .*

INPUT: A digital image  $(\mathcal{V}, I, 14, 14)$ , a point  $v \in I$  and an AM-model  $(C_I, \phi_I)$  for  $\mathcal{I}$  such that in each dimension  $q$ , the matrix of  $\partial_q$  with respect to  $C_I$  coincides with its SNF.



Let  $\{\sigma_1, \dots, \sigma_n\}$  ( $n \leq 74$ ) be the ordered-by-decreasing-dimension set of all the simplices of  $K(I)$  having  $v$  as a vertex.

$K := K(I)$ ,  $C^{\setminus v} := C_I$  and  $\phi^{\setminus v} := \phi_I$ .

For  $i = 1$  to  $i = n$  do

Let  $q$  be the dimension of  $\sigma_i$ , let  $C_q^{\setminus v} = \{a_1, \dots, a_r\}$  and  $C_{q-1}^{\setminus v} = \{e_1, \dots, e_s\}$  such that  $\partial_q(a_j) = e_j$  for  $1 \leq j \leq t$ ,  $\partial_q(a_j) = 0$  for  $t < j \leq r$  and  $\partial_q(\sigma_i) = \sum_{\ell=1}^s \rho_\ell e_\ell$  for some  $\rho_\ell \in \mathbf{Z}$ .  
Let  $k$  be the smallest index such that  $\{a_1, \dots, \hat{a}_k, \dots, a_r\}$  is a base of  $C_q(K \setminus \{\sigma_i\})$  then

$C_q^{\setminus v} := \{a_1, \dots, \hat{a}_k, \dots, a_r\}$  and  $\phi(e_k) := 0$ .

$K := K \setminus \{\sigma_i\}$ .

OUTPUT: An AM-model  $(C^{\setminus v}, \phi^{\setminus v})$  for  $\mathcal{I}^{\setminus v}$ .

Observe that if an AM-model  $(C, \phi)$  for  $\mathcal{I}$  is computed using Algorithm 2.2, 4.1 or 4.2 then in each dimension  $q$ , it satisfies that the matrix of  $\partial_q$  with respect to  $C$  coincides with its SNF.

#### 4.3. AM-models for 3D Digital Images under Voxel-Set Operations

In this subsection, we reuse the AM-model information for digital images under voxel-set operations (union, intersection, difference and inverse). Let  $\mathcal{I} = (\mathcal{V}, I, 14, 14)$  and  $\mathcal{J} = (\mathcal{V}, J, 14, 14)$  be two digital images, then  $\mathcal{I} \cup \mathcal{J} = (\mathcal{V}, I \cup J, 14, 14)$ ,  $\mathcal{I} \cap \mathcal{J} = (\mathcal{V}, I \cap J, 14, 14)$ ,  $\mathcal{I} \setminus \mathcal{J} = (\mathcal{V}, I \setminus J, 14, 14)$ . In order to define the inverse of  $\mathcal{I}$ , for each  $p = (x_1, x_2, x_3) \in \mathcal{V}$ , let  $X_p = \max\{|x_i| : i = 1, 2, 3\}$ . Let  $X_I = \max\{X_p : p \in I\}$ . Let  $\mathcal{G}_I$  be the digital image  $(\mathcal{V}, G_I, 14, 14)$  where  $G_I = \{p : p \in \mathcal{V} \text{ and } X_p \leq X_I + 1\}$ . Then, the inverse of  $\mathcal{I}$ ,  $\bar{\mathcal{I}}$ , is  $\mathcal{G}_I \setminus \mathcal{I}$ . We will not consider any of the trivial cases  $I = \emptyset$ ,  $J = \emptyset$ ,  $I \cap J = \emptyset$ ,  $I \subseteq J$ , or  $I = \{p : p \in \mathcal{V} \text{ and } X_p \leq r\}$  for some  $r \in \mathbf{Z}$ .

Let  $\mathcal{L} = (\mathcal{V}, L, 14, 14)$  be a digital image and  $F = \{v_1, \dots, v_m\} \subset \mathcal{V}$  such that  $F \subset L$  or  $F \cap L = \emptyset$ . If  $F \subset L$ , denote by  $\mathcal{L}^F$  the image  $(\mathcal{V}, L \setminus F, 14, 14)$ . On the other hand, if  $F \cap L = \emptyset$ , denote by  $\mathcal{L}^F$  the image  $(\mathcal{V}, L \cup F, 14, 14)$ . Let  $(C_L, \phi_L)$  be an AM-model for  $\mathcal{L}$  such that in each dimension  $q$ , the matrix of  $\partial_q$  with respect to  $C_L$  coincides with its SNF. Algorithm 4.3 is a common processing to the four voxel-set operations treated here.

#### Algorithm 4.3. Common Processing.

INPUT: The AM-model  $(C_L, \phi_L)$  for  $\mathcal{L} = (\mathcal{V}, L, 14, 14)$  and the set of points  $F = \{v_1, \dots, v_m\}$  such that  $F \subset L$  or  $F \cap L = \emptyset$ .

If  $F \subset L$  then  
   For  $i = 1$  to  $i = m$  do  
     apply Algorithm 4.2 to  $v_i$  and the AM-model  $(C_L, \phi_L)$ .  
      $C_L := C_L^{\setminus v_i}$  and  $\phi_L := \phi_L^{\setminus v_i}$ .  
 Else for  $i = 1$  to  $i = m$  do  
   apply Algorithm 4.1 to  $v_i$  and the AM-model  $(C_L, \phi_L)$ .  
    $C_L := C_L^{\cup v_i}$  and  $\phi_L := \phi_L^{\cup v_i}$ .  
 $C_L^F := C_L$  and  $\phi_L^F := \phi_L$   
 OUTPUT: An AM-model  $(C_L^F, \phi_L^F)$  for  $\mathcal{L}^F$ .

Let  $(C_I, \phi_I)$  and  $(C_J, \phi_J)$  be an AM-model for  $\mathcal{I}$  and  $\mathcal{J}$ , respectively, such that in each dimension  $q$ , the matrix of  $\partial_q$  with respect to  $C_L$ , for  $L = I, J$ , coincides with its SNF. The following algorithm computes an AM-model for  $\mathcal{I} \cup \mathcal{J}$ .

**Algorithm 4.4.** *Computing an AM-model for  $\mathcal{I} \cup \mathcal{J}$ .*

INPUT: The AM-models  $(C_I, \phi_I)$  for  $\mathcal{I}$  and  $(C_J, \phi_J)$  for  $\mathcal{J}$ .  
 Apply Algorithm 4.3 to  $(C_L, \phi_L)$  and  $F := I \cap J$  for  $L = I, J$ .  
 For each  $a \in C := C_I^F \cup C_J^F$   
    $\phi(a) := \phi_I^F(a)$  if  $a \in C_I^F$ ; and  $\phi(a) := \phi_J^F(a)$  if  $a \in C_J^F$ .  
 Apply Algorithm 4.3 to  $F$  and the AM-model  $(C, \phi)$ .  
 $C_{I \cup J} := C^F$  and  $\phi_{I \cup J} := \phi^F$ .  
 OUTPUT: An AM-model  $(C_{I \cup J}, \phi_{I \cup J})$  for  $\mathcal{I} \cup \mathcal{J}$ .

Algorithm 4.3 is the essential step for computing an AM-model for  $\mathcal{I} \cap \mathcal{J}$ ,  $\mathcal{I} \setminus \mathcal{J}$  and  $\mathcal{I}$ .

**Algorithm 4.5.** *Computing an AM-model for  $\mathcal{I} \cap \mathcal{J}$ .*

INPUT: The AM-model  $(C_I, \phi_I)$  for  $\mathcal{I}$  and the set  $F = I \setminus J$ .  
 Apply Algorithm 4.3 to  $(C_I, \phi_I)$  and  $F$ .  
 OUTPUT: An AM-model  $(C_I^F, \phi_I^F)$  for  $\mathcal{I} \cap \bar{\mathcal{J}}$ .

An algorithm for computing an AM-model for  $\mathcal{I} \setminus \mathcal{J}$  (resp. for  $\bar{\mathcal{I}}$ ) is similar to the one above. The only difference is that the input is an AM-model  $(C_I, \phi_I)$  for  $\mathcal{I}$  and the set  $F = I \cap J$  (resp. an AM-model  $(C_{G_I}, \phi_{G_I})$  for  $\mathcal{G}_I$  and the set  $I$ ).

## 5. Comments and Future Work

The (non-unique) algebraic-topological representation of a given simplicial complex of any dimension showed here, allows to compute integer (co)homology, representative cycles of integer (co)homology generators, the cup product on cohomology with integer coefficients and a topological invariant derived from the integer cohomology ring. Moreover, we give a positive answer to the problem of reusing AM-models for determining homological information of new 3D binary digital images constructed from the previous ones using voxel-set operations.

There is considerable scope for further research:

- To extend our method to  $n$ D binary digital images in any grid using simplicial analogous techniques [14, 13, 3].
- To compute topological invariants from primary cohomology operations in the discrete setting of digital images. The work done in [8] seems to be a compulsory reading for advancing in this issue.
- To compute homotopy groups of  $n$ D binary digital images or  $n$ - $G$ -maps [19] using AM-models. The works [16, 14, 21, 23, 3] could help us in this task.
- To develop a discrete Morse theory [7] for digital images well-adapted to our method. The paper [22] would be a good starting point.

Potential applications of our method in Computer Vision and Digital Image Processing involving not only 3D object but also higher dimensional structures can be encountered in Medical Imaging and Object Modelling. Moreover, our method seems to be especially well-adapted to segmentation under topological constraints and elimination of small topological noise.

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